Chapman-Kolmogorov theorem.

This theorem, so far only vaguely known to me, is useful in relation to multi-step probability computations in random walks, queuing theory and statistical physics. It is a fundamental property of **Markov processes** (both discrete and continuous time) that expresses the relationship between **transition probabilities** over different time intervals.

Formally, if Pij(t) is the probability that a process in state i at time 0 is in state j at time t, then:

$$P_{ij}(t+s) = \sum_k P_{ik}(t)\, P_{kj}(s)$$

Here:

- i,j,k are **states**.
- t,s are **time intervals**.
- The sum is over all **possible intermediate states** k.
  The probability of going from state i to j in total time t+s is the sum over all possible intermediate states k of: Probability of going from i to k in

Let's take a discrete-time Markov chain with 3 states: A,B,C.
Transition matrix for one step: P(1) probabilities going from A to A, A to B, A to C, etc
Transition matrix for two steps P(2) can be calculated by multiplying the matrix by itself

$$P(1) = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.4 \end{bmatrix}$$

AA AB AC
BA BB BC
CA CB CC

$$P(2)) = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} \mathbf{0.3} & \mathbf{0.3} & \mathbf{0.2} \\ & & \\ & & \end{bmatrix}$$

Here, $P_{AB}(1) = 0.3$ means: if we're in A now, there's a 30% chance of being in B next step.

Now, what's $P_{AC}(2)$ (two steps from A to C)?

Using Chapman–Kolmogorov:

$$P_{AC}(2) = P_{AA}(1)P_{AC}(1) + P_{AB}(1)P_{BC}(1) + P_{AC}(1)P_{CC}(1)$$

Plugging in:

$$P_{AC}(2) = (0.5)(0.2) + (0.3)(0.2) + (0.2)(0.4)$$

$$P_{AC}(2) = 0.10 + 0.06 + 0.08 = 0.24$$

This matches exactly what you'd get by **multiplying the matrix by itself**:

$$P(2) = P(1) \times P(1)$$

The Chapman–Kolmogorov theorem is just this principle generalized to **any** time intervals.

✅ **Summary:**

- **Statement:** $P_{ij}(t+s) = \sum_k P_{ik}(t) P_{kj}(s)$
- **Use:** Fundamental for multi-step transitions, Markov chain analysis, and building stochastic process theory.
- **Intuition:** Probability of getting from i to j in $t+s$ equals "all possible intermediate routes" via some k.

Simulating the Markov process and displayed a table comparing the **empirical two-step probabilities** with the **theoretical values** from the Chapman–Kolmogorov theorem shows an excellent approximation:

	From	To	P2_empirical	P2_theoretical
0	A	A	0.39011	0.39
1	A	B	0.36775	0.37
2	A	C	0.24214	0.24
3	B	A	NaN	0.38
4	B	B	NaN	0.38